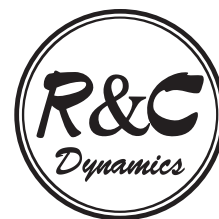


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DYNAMICS OF ROLLING DISK

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In the paper we present the qualitative analysis of rolling motion without slipping of a homogeneous round disk on a horizontal plane. The problem was studied by S. A. Chaplygin, P. Appel and D. Korteweg who showed its integrability. The behavior of the point of contact on a plane is investigated and conditions under which its trajectory is finite are obtained. The bifurcation diagrams are constructed.

1. Introduction

For the first time the motion of a heavy dynamically symmetrical round disk on a horizontal absolutely rough plane was investigated by G Slesser (1861) [28], N. Ferrers (1872) [9], K. Neumann (1886), and A. Firkandt (1892). These studies eventually (after unsuccessful attempts by Neumann and Lindelöf) lead to the correct form of equations of motion. This form differs from the usual (Lagrangian or Hamiltonian) equations of mechanics because of the nonholonomic constraint showing that the velocity of the point of contact of a disk with a plane is zero. We shall not discuss in detail the general forms of the equations of the nonholonomic mechanics (they are presented, for example, in [23], [26]). Instead, we concentrate on the pretty obvious form of these equations obtained from the general principle of dynamics — the conservation law of the moment of momentum written in the disk-fixed axes.

S. A. Chaplygin (1897) was the first to show the integrability of the problem on rolling motion of a disk. He presented the reduction of the problem to the analysis of hypergeometric quadratures in paper [6], where he showed also the integrability of the problem on rolling motion of an arbitrary heavy dynamically symmetric body of rotation on a horizontal plane — in the latter case the problem is reduced to the integration of the linear differential second-order equation. The integration of equations of motion of a disk in hyperelliptic functions was also performed in 1900 independently from each other and from Chaplygin by P. Appel [2] and D. Korteweg [14]. Sometimes the problem on rolling motion of a disk is referred to as Appel–Korteweg problem (or simply Appel problem), but this is, probably, not quite correct. In 1903 the same result has been rediscovered by E. Gellop [10], however he used the Legendre functions.

Despite of the explicit hypergeometric quadratures the various qualitative properties of disk motion were not studied for the long time. There were mainly studies of stationary motions and of their stability (the corresponding bibliography is presented in book [23]). Some qualitative properties of the disk motion have been discussed only in papers S. N. Kolesnikov [13] and Yu. N. Fedorov [8]. The first paper shows that for almost all initial conditions the disk never falls onto a plane and the second one presents the procedure of investigation of the reduced system. Analogous results for the dynamically asymmetrical disk and disk moving on an inclined plane (nonintegrable problems) were obtained in [1], [18]. Among the modern works analyzing the rolling motion of the disk we shall note papers O. M. O'Reilly [27], R. Cushman, J. Hermans, D. Kemppainen [7], and

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A.S. Kuleshov [21] devoted to the study of bifurcations and stability of stationary motions of the disk.

General results of a qualitative analysis for the rolling motion of a heavy body of rotation were obtained in paper N.K. Moshuk [24]. The paper include the frequency analysis, application of the KAM-theory, and basic qualitative properties for the motion of the point of contact. It appears that the point of contact performs the composite bounded motion: it periodically traces some closed curve which rotates as a rigid body with some constant angular velocity about the fixed point. Thus the realization of some resonance relation between frequencies makes possible the drift of the body of rotation to the infinity.

In this paper we develop these qualitative considerations and complement them with the computer analysis. We also present various types of trajectories which are traced by the point of contact in the body-fixed and relative frames of references since they have curious forms which are difficult to predict. Using the computer modelling we explicitly investigate the hypothesis about the drift to the infinity under the resonance conditions. We present the most general three-dimensional bifurcation diagram in the space of the first integrals and the complete atlas of its sections by various planes, constructed with the help of computer modelling.

In this paper we also present a new method of reduction of the problem to an one-degree integrable Hamiltonian system and explicitly consider the existence of Hamiltonian formulation for different variants of equations of motion of the problem.

2. The rolling motion of a rigid body on a plane

2.1. Equations of motion and their integrals

Let the rigid body in an exterior field of force perform a rolling motion on a plane without sliding. In this case the equations of motion have the most convenient form in the body-fixed frame of references which axes are directed along the principal axes of inertia of the body and the origin is situated at the center of mass. In the following text all vectors are assumed to be projected on these axes.

The condition of absence of slipping thus becomes

$$\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} = 0, \quad (2.1)$$

where \mathbf{v} , $\boldsymbol{\omega}$ are the velocity of the center of mass and the angular velocity of the body and \mathbf{r} is the vector directed from the center of mass to the point of contact (see fig. 1).

Let's denote the projections of the fixed basis vectors to the moving axes by α , β , γ (the vector γ is perpendicular to the plane) and by (x, y) we shall denote the coordinates of the projection of the center of mass onto the plane in the fixed frame of references. We

assume that the field of force is potential with a potential depending

only on the orientation of the body $U = U(\alpha, \beta, \gamma)$. The complete set of the equations of motion defining the given system can be represented in the form

$$\dot{\mathbf{M}} = \mathbf{M} \times \boldsymbol{\omega} + m \dot{\mathbf{r}} \times (\boldsymbol{\omega} \times \mathbf{r}) + \alpha \times \frac{\partial U}{\partial \alpha} + \beta \times \frac{\partial U}{\partial \beta} + \gamma \times \frac{\partial U}{\partial \gamma}, \quad (2.2)$$

$$\dot{\alpha} = \alpha \times \boldsymbol{\omega}, \quad \dot{\beta} = \beta \times \boldsymbol{\omega}, \quad \dot{\gamma} = \gamma \times \boldsymbol{\omega}. \quad (2.3)$$

The equation (2.2) describes the evolution of the vector of moment of momentum for the body with respect to the point of contact \mathbf{M} and (2.3) concerns the evolution of the fixed basis vectors in the body-fixed frame of references.

The motion of the center of mass can be obtained in quadratures from solutions of the equations (2.2), (2.3) as follows

$$\dot{x} = (\mathbf{r} \times \boldsymbol{\omega}, \boldsymbol{\alpha}), \quad \dot{y} = (\mathbf{r} \times \boldsymbol{\omega}, \boldsymbol{\beta}). \quad (2.4)$$

The expression of the vector of moment of momentum with respect to the point of contact \mathbf{M} can be written in the following form

$$\mathbf{M} = \mathbf{I}\boldsymbol{\omega} + m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}), \quad (2.5)$$

where $\mathbf{I} = \text{diag}(I_1, I_2, I_3)$ is the tensor of inertia of the body. In turn \mathbf{r} can be uniquely expressed (for a convex body) through the normal to the plane γ from the equation

$$\gamma = -\frac{\nabla F(\mathbf{r})}{|\nabla F(\mathbf{r})|}, \quad (2.6)$$

Here $F(\mathbf{r}) = 0$ is the equation of the body's surface.

Let's consider a motion of the point of contact on a plane. If we denote the position of the point of contact on the plane in the fixed frame of references as (X, Y) , then the equation of motion for the point of contact can be presented in the form

$$\dot{X} = (\dot{\mathbf{r}}, \boldsymbol{\alpha}), \quad \dot{Y} = (\dot{\mathbf{r}}, \boldsymbol{\beta}). \quad (2.7)$$

where $\dot{\mathbf{r}}$ is determined from equations (2.2) – (2.6). Actually \dot{X} and \dot{Y} are projections of the velocity of the point of contact in the relative frame of reference onto the fixed axes.

The equations of motion in the form similar to (2.2) – (2.3) are presented, for example, in book [11]. They can be obtained also by means of Poincaré–Chetaev formalism [3] with undetermined Lagrangian coefficients; these coefficients shall be eliminated with the help of the constraints' equations (2.1).

The system (2.2) – (2.3) generally has seven independent integrals of motion, six of them are trivial geometrical integrals:

$$\begin{aligned} \alpha^2 &= 1, & \beta^2 &= 1, & \gamma^2 &= 1, \\ (\alpha, \beta) &= 0, & (\beta, \gamma) &= 0, & (\gamma, \alpha) &= 0. \end{aligned} \quad (2.8)$$

The seventh is the integral of energy

$$\frac{1}{2}(\mathbf{M}, \boldsymbol{\omega}) + U(\alpha, \beta, \gamma) = h = \text{const}. \quad (2.9)$$

Generally the given system has no other additional integrals and the possibility of its integrability in concrete cases depends on the presence of additional tensor invariants (measure, fields of symmetry, integrals).

2.2. The rolling motion of a heavy disk

Let's consider the case of rolling motion for an axially symmetric disk of radius R in the field of gravity. The field is, obviously, also axially symmetric with the potential depending only on γ . Moreover, we suppose that the disk is dynamically symmetric, i. e. $I_1 = I_2$. The potential energy in this case has the following form

$$U = -mg(\mathbf{r}, \gamma) = mgR\sqrt{1 - \gamma_3^2}. \quad (2.10)$$

The equation of surface for the disk is $F(\mathbf{r}) = r_1^2 + r_2^2 - R^2$. Substituting it in the equation (2.6) and solving with respect to \mathbf{r} we obtain

$$r_1 = -\frac{R\gamma_1}{\sqrt{1 - \gamma_3^2}}, \quad r_2 = -\frac{R\gamma_2}{\sqrt{1 - \gamma_3^2}}, \quad r_3 = 0. \quad (2.11)$$

As the potential energy depends only on γ , in the equations of motion (2.2) – (2.3) we get the separate system of six equations.

$$\begin{aligned}\dot{\mathbf{M}} &= \mathbf{M} \times \boldsymbol{\omega} + m\dot{\mathbf{r}} \times (\boldsymbol{\omega} \times \mathbf{r}) + mg\mathbf{r} \times \boldsymbol{\gamma}, \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \boldsymbol{\omega}.\end{aligned}\quad (2.12)$$

Expressing $\boldsymbol{\omega}$, \mathbf{r} from relations (2.5), (2.11) we shall get the closed system for the variables \mathbf{M} , $\boldsymbol{\gamma}$ similar in many aspects to the Euler–Poisson system in the Lagrange case, however the obtained system is much more complicated than the last one.

The equations (2.12) preserve the geometrical integral γ^2 and the energy (2.9), in addition they allows the standard invariant measure (with a constant density). For the integrability (by Euler–Jacobi [17]) of these equations we need two additional integrals. In the following we describe the method of derivation of these integrals.

The possibility of separation of the system (2.12) from the general system (2.2) – (2.3) is connected to the symmetry with respect to the rotations about the vertical axis defined by the vector $\boldsymbol{\gamma}$. The system (2.12) is invariant with respect to the field of symmetries commuting with the vector field of the problem.

$$\widehat{\mathbf{v}}_\psi = \alpha_1 \frac{\partial}{\partial \beta_1} - \beta_1 \frac{\partial}{\partial \alpha_1} + \alpha_2 \frac{\partial}{\partial \beta_2} - \beta_2 \frac{\partial}{\partial \alpha_2} + \alpha_3 \frac{\partial}{\partial \beta_3} - \beta_3 \frac{\partial}{\partial \alpha_3}, \quad (2.13)$$

It is possible to show that the variables \mathbf{M} , $\boldsymbol{\gamma}$ are the integrals of field (2.13) that is $\widehat{\mathbf{v}}_\psi(M_i) = 0$, $\widehat{\mathbf{v}}_\psi(\gamma_i) = 0$, $i = 1, 2, 3$. According to the general Lie theory [19], variables \mathbf{M} , $\boldsymbol{\gamma}$ define the reduced system. For the classical Euler–Poisson equations the corresponding reduction is the Raus reduction with respect to the cyclical angle of precession.

In addition to the field of symmetries (2.13) the equations of motion (2.2) – (2.3) for the axially symmetric body allow one more field of symmetries corresponding to the rotation about the axis of symmetry of the disk.

$$\begin{aligned}\widehat{\mathbf{v}}_\varphi &= M_1 \frac{\partial}{\partial M_2} - M_2 \frac{\partial}{\partial M_1} + \gamma_1 \frac{\partial}{\partial \gamma_2} - \gamma_2 \frac{\partial}{\partial \gamma_1} + \\ &+ \alpha_1 \frac{\partial}{\partial \alpha_2} - \alpha_2 \frac{\partial}{\partial \alpha_1} + \beta_1 \frac{\partial}{\partial \beta_2} - \beta_2 \frac{\partial}{\partial \beta_1}.\end{aligned}\quad (2.14)$$

It is possible to show that integrals of the field (2.14) are projections of the moment and normal to the plane of disk onto the fixed axes of coordinates

$$\mathbf{N} = ((\mathbf{M}, \boldsymbol{\alpha}), (\mathbf{M}, \boldsymbol{\beta}), (\mathbf{M}, \boldsymbol{\gamma})), \quad \mathbf{n} = (\alpha_3, \beta_3, \gamma_3).$$

The equations of motion for these variables can be presented in the following form

$$\begin{aligned}\dot{\mathbf{N}} &= m\tilde{\mathbf{r}} \times (\tilde{\boldsymbol{\omega}} \times \tilde{\mathbf{r}}) + mg\tilde{\mathbf{r}} \times \mathbf{n}, \\ \dot{\mathbf{n}} &= \tilde{\boldsymbol{\omega}} \times \mathbf{n},\end{aligned}\quad (2.15)$$

where symbols $\tilde{\boldsymbol{\omega}}$, $\tilde{\mathbf{r}}$ denote the same vectors, but projected onto the fixed axes (that is $\tilde{\boldsymbol{\omega}}_1 = (\boldsymbol{\omega}, \boldsymbol{\alpha}), \dots, \tilde{\mathbf{r}}_1 = (\mathbf{r}, \boldsymbol{\alpha}), \dots$). The explicit expression of the components of the vector $\tilde{\mathbf{r}}$ is

$$\tilde{\mathbf{r}} = \left(\frac{R\alpha_3\gamma_3}{\sqrt{1-\gamma_3^2}}, \frac{R\beta_3\gamma_3}{\sqrt{1-\gamma_3^2}}, -R\sqrt{1-\gamma_3^2} \right). \quad (2.16)$$

The vector \mathbf{N} is expressed through $\boldsymbol{\omega}$ by the formula

$$\mathbf{N} = I_1\tilde{\boldsymbol{\omega}} + (I_3 - I_1)(\tilde{\boldsymbol{\omega}}, \mathbf{n})\mathbf{n} + m\tilde{\mathbf{r}} \times (\tilde{\mathbf{r}} \times \tilde{\boldsymbol{\omega}}). \quad (2.17)$$

REMARK 1. Such reduction is also possible for an arbitrary body of rotation.

2.3. A reduction to the integrable one-degree Hamiltonian system

Let's describe the process of reduction of order with respect to the both fields of symmetries (2.13) and (2.14). For that we shall choose the simultaneous integrals of these fields as variables of the reduced system. According to [5], the most convenient algebraic set of such variables is

$$\begin{aligned}\gamma_3, \quad K_1 &= M_1\gamma_1 + M_2\gamma_2 = N_3 - \gamma_3(\mathbf{N}, \mathbf{n}), \\ K_2 &= \sqrt{\frac{I_1}{I_3 + mR^2}} M_3 = \sqrt{\frac{I_1}{I_3 + mR^2}} (\mathbf{N}, \mathbf{n}), \\ K_3 &= \gamma_1 M_2 - \gamma_2 M_1 = N_1 n_2 - N_2 n_1.\end{aligned}\tag{2.18}$$

The equations of motion in the new variables become

$$\begin{aligned}\dot{\gamma}_3 &= \frac{K_3}{I_1 + mR^2}, \\ \dot{K}_1 &= -\frac{I_3}{(I_1 + mR^2)\sqrt{I_1(I_3 + mR^2)}} K_3 K_2, \\ \dot{K}_2 &= -\frac{mR^2}{(I_1 + mR^2)\sqrt{I_1(I_3 + mR^2)}} \frac{K_3 K_1}{1 - \gamma_3^2}, \\ \dot{K}_3 &= -\frac{\gamma_3}{1 - \gamma_3^2} \left(\frac{K_1^2}{I_1} + \frac{K_2^2}{I_1 + mR^2} \right) + \\ &+ \frac{\sqrt{I_1(I_3 + mR^2)}}{I_1^2} K_1 K_2 + mgR\gamma_3 \sqrt{1 - \gamma_3^2}.\end{aligned}\tag{2.19}$$

The equations (2.19) preserve the invariant measure with density $\rho = \frac{1}{1 - \gamma_3^2}$. Dividing the second and the third equations on the first and choosing a new independent variable, the angle of nutation $\theta = \arccos \gamma_3$, we shall get the system of linear equations

$$\frac{dK_1}{d\theta} = \frac{I_3 \sin \theta}{\sqrt{I_1(I_3 + mR^2)}} K_2, \quad \frac{dK_2}{d\theta} = \frac{mR^2}{\sqrt{I_1(I_3 + mR^2)}} \frac{K_1}{\sin \theta}.\tag{2.20}$$

The general solution of these equations can be presented in the form [23]

$$\begin{aligned}K_1 &= C_1 \frac{I_3 \sin^2 \theta}{2\sqrt{I_1(I_3 + mR^2)}} F(1 + \xi, 1 + \eta, 2, \frac{1 - \cos \theta}{2}) - \\ &- C_2 \frac{I_3 \sin^2 \theta}{2\sqrt{I_1(I_3 + mR^2)}} F(1 + \xi, 1 + \eta, 2, \frac{1 + \cos \theta}{2}), \\ K_2 &= C_1 F(\xi, \eta, 1, \frac{1 - \cos \theta}{2}) + C_2 F(\xi, \eta, 1, \frac{1 + \cos \theta}{2}),\end{aligned}\tag{2.21}$$

where ξ and η are the solutions of the quadratic equation $x^2 - x + \frac{I_3 m R^2}{I_1(I_3 + mR^2)} = 0$ and $F(\xi, \eta, n, z)$ is the generalized hypergeometric function representable by series

$$F(\xi, \eta, n, z) = \sum_{k=0}^{\infty} \frac{\Gamma(\xi + k)\Gamma(\eta + k)\Gamma(n)}{\Gamma(\xi)\Gamma(\eta)\Gamma(n + k)} \frac{z^k}{k!}\tag{2.22}$$

Thus, the relations (2.21) define (implicitly) the integrals of motion. In this case they are the “constants” C_1 and C_2 expressed through K_1 , K_2 , θ .

The quadrature for the angle of nutation can be obtained from the integral of energy written in the variables K_1, K_2, K_3, θ

$$\begin{aligned}\dot{\theta}^2 &= 2 \sin^2 \theta (I_1 + mR^2) P(\theta), \\ P(\theta) &= h - \frac{K_1^2}{2I_1 \sin^2 \theta} - \frac{1}{2} \frac{K_2^2}{I_1} - mgR \sin \theta.\end{aligned}\quad (2.23)$$

Here we assume that the variables K_1, K_2 are expressed through the constants of integrals and angle θ according to the formulas (2.21). In this case the function $P(\theta)$ (depending on the constants of integrals) define the analog of gyroscopic function for the Lagrange top [3], [22].

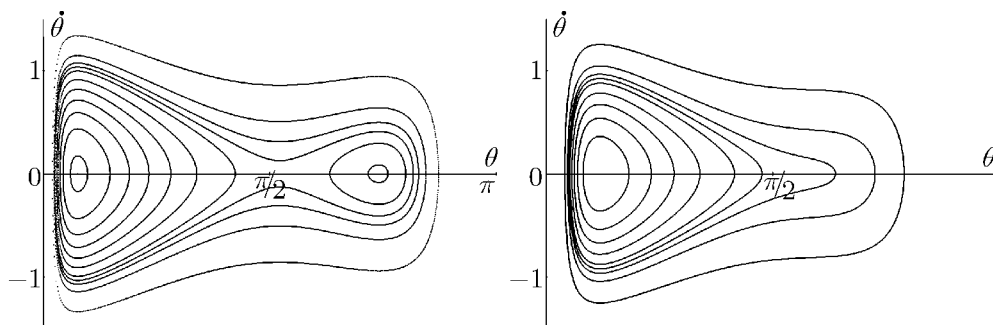


Fig. 2. Phase portraits of the system (2.23) at various values C_1 and C_2 . Left: the case of existence of three periodic solutions ($C_1 = 0.05, C_2 = 0.01$). Right: the case of existence of one periodic solution ($C_1 = 0.08, C_2 = -0.02$).

Thus, the equation (2.23) at the fixed values C_1 and C_2 define the one-degree Hamiltonian system. The phase portraits of this system on the plane $\theta, \dot{\theta}$ are presented in fig. 2. All the variables γ_3, K_1, K_2, K_3 are periodic functions of time with the period T_θ and the corresponding frequency ω_θ .

REMARK 2. According to [5], the system (2.19) is the Hamiltonian one with degenerated Poisson bracket which has two Casimir functions expressed through hypergeometric functions.

2.4. Quadratures for angles of proper rotation and a precession

According to the general Lie theory [19], if the variables of reduced system (2.18) are the given functions of time, then all the variables of initial system (2.12) (and accordingly (2.15)) can be obtained by one quadrature (if fields \hat{v}_ψ (2.13) and \hat{v}_ϕ (2.14) are commuting).

Indeed, using the equalities $\operatorname{tg} \varphi = \frac{\gamma_1}{\gamma_2}$ (and correspondingly $\operatorname{tg} \psi = -\frac{n_1}{n_2}$) for angles φ and ψ , we obtain

$$\dot{\varphi} = -\frac{\gamma_3}{1 - \gamma_3^2} \frac{K_1}{I_1} + \frac{K_2}{\sqrt{I_1(I_3 + mR^2)}}, \quad \dot{\psi} = -\frac{K_1}{I_1(1 - \gamma_3^2)}.\quad (2.24)$$

Thus, for each of the angles the dependence on time is defined as an integral of a periodic function with the frequency ω_θ , hence it can be presented in the form (see, for example, [17], [24])

$$\varphi = \omega_\varphi t + \varphi_*(t), \quad \psi = \omega_\psi t + \psi_*(t),\quad (2.25)$$

where $\varphi_*(t), \psi_*(t)$ are periodic function with frequency ω_θ . Moreover, (2.24) and (2.25) imply also that all the frequencies $\omega_\theta, \omega_\varphi, \omega_\psi$ depend only on the constants of the first integrals.

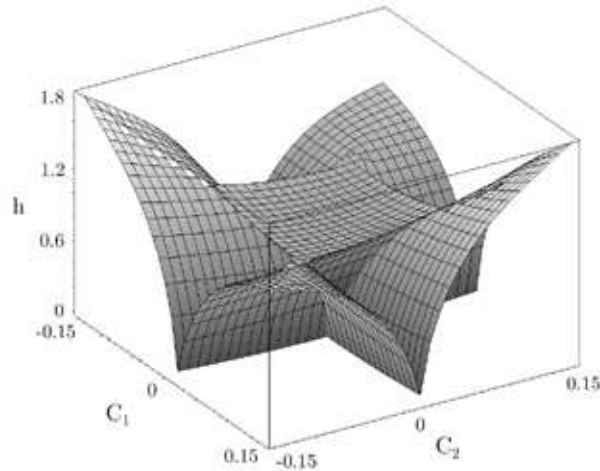


Fig. 3. The surface of regular precessions. Parameters of the system are $I_1 = 0.25$, $I_2 = 0.5$, $R = 1$, $m = 1$, $g = 1$.

2.5. A motion of the point of contact

Following papers [20], [24] we present the equation for the velocity of the point of contact in the form

$$\dot{Z} = R \left(\frac{\gamma_3}{1 - \gamma_3^2} \frac{K_1}{I_1} - \frac{K_2}{\sqrt{I_1(I_3 + mR^2)}} \right) e^{i\psi}, \quad (2.26)$$

where $Z = X + iY$ and X , Y are the coordinates of the point of contact in the fixed frame of references.

Thus the coordinates of the point of contact are determined by quadratures of quasiperiodic two-frequency (with the frequencies ω_ψ , ω_θ) functions of time.

3. The qualitative analysis and results

Let's perform the qualitative analysis of the dynamics of the disk motion. We will make a classification of all possible motions depending on the constants of the first integrals. Some features of the considered case essentially complicate this work in comparison with the case of the Lagrange top for Euler–Poisson equations. For uniformity we recommend to study such analysis for the Lagrange case in book [3]. The complexity of analysis is caused by the facts that the integrals of motion can not be expressed in elementary functions (only in special one) and the system has no natural Hamiltonian presentation. Moreover, in addition to the motion of apexes of the body (disk) we shall classify trajectories of the point of contact obtained by additional quadratures of quasiperiodic functions.

3.1. The bifurcation analysis of the reduced system

Possible types of motion for the axis of symmetry of the body are completely determined by the form of the gyroscopic function $P(\theta)$ and by the energy level. Critical values of the integrals of motion C_1 , C_2 , h are determined by the equations

$$P(\theta) = 0, \quad \frac{dP(\theta)}{d\theta} = 0. \quad (3.1)$$

In three-dimensional space with coordinates C_1 , C_2 , h equations (3.1) define a three-dimensional surface, so-called *surface of regular precessions* [3] (see fig. 3). This name is connected to the fact that at the given values of integrals the coin performs motion with the fixed angle $\theta = \text{const}$, which is analogous to the precession for Lagrange top [22]. The full atlas of sections of the surface of regular precessions (bifurcation diagrams) by planes $C_1 + C_2 = \text{const}$ and $C_1 - C_2 = \text{const}$ is presented in

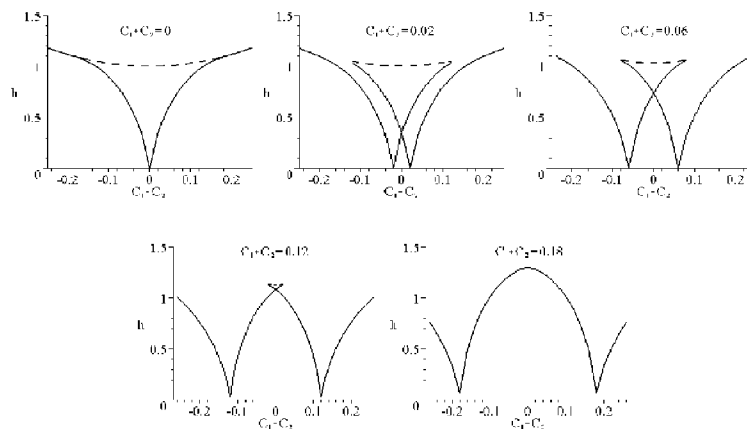


Fig. 4. Sections of the surface of regular precessions represented in fig. 3 by planes $C_1 + C_2 = \text{const}$.

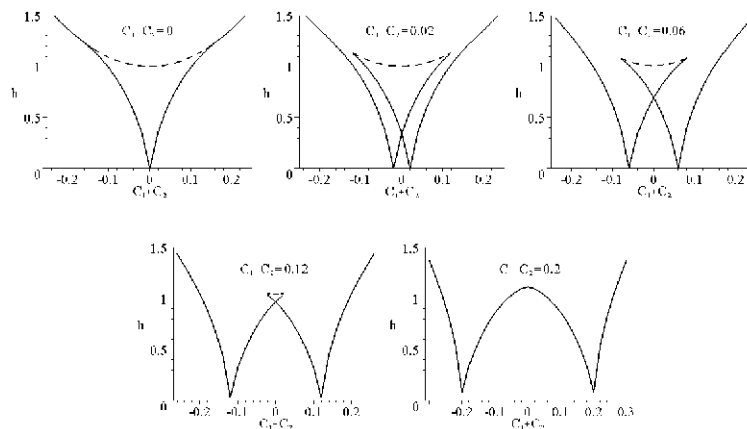


Fig. 5. Sections of the surface of regular precessions represented in fig. 3 by planes $C_1 - C_2 = \text{const}$.

figs. 4 and 5 accordingly. In fig. 6 and 7 for two different sections we show the forms of the gyroscopic function $P(\theta)$ corresponding to various values of integrals C_1 , C_2 , h .

Using these figures (and the rule of signs) we can easily study the stability of the corresponding solutions located on branches of the bifurcation diagram (branches corresponding to unstable solutions are represented on the diagram by a dotted line). In figs. 6 and 7 vertical straight lines represent cases when $C_1 = 0$ or $C_2 = 0$. In these cases the disk motion corresponds to the falling and planes determined by these equalities define in space of integrals C_1 , C_2 , h the two-dimensional manifold of fallings. Thus for almost all initial conditions the disk do not fall performing the rolling motion on a plane.

Other remarkable motions correspond to the cases $C_1 = C_2$, the rolling motion of the disk, and $C_1 = -C_2$, the rotation of the disk about its axis passing through the diameter. During the latter motion the declination of the disk with respect to the vertical remains constant.

REMARK 3. The bifurcation diagram (fig. 3, 4, 5) is different from one presented in papers [21], [27] since instead of the value of energy we use the value of angle of declination corresponding to the precession θ_0 and this function has no physical sense for other motions (when this angle is not preserved). Only the points on the surface of regular precessions have the physical sense. At the same time each value of constants C_1 , C_2 , h in space of integrals in fig. 3 corresponds to some motion whether this point is situated on the surface of regular precessions or not and this is important for the qualitative analysis.

REMARK 4. One of sections of the three-dimensional diagram by a plane $h = \text{const}$ and the corresponding gyroscopic functions are presented in paper [7].

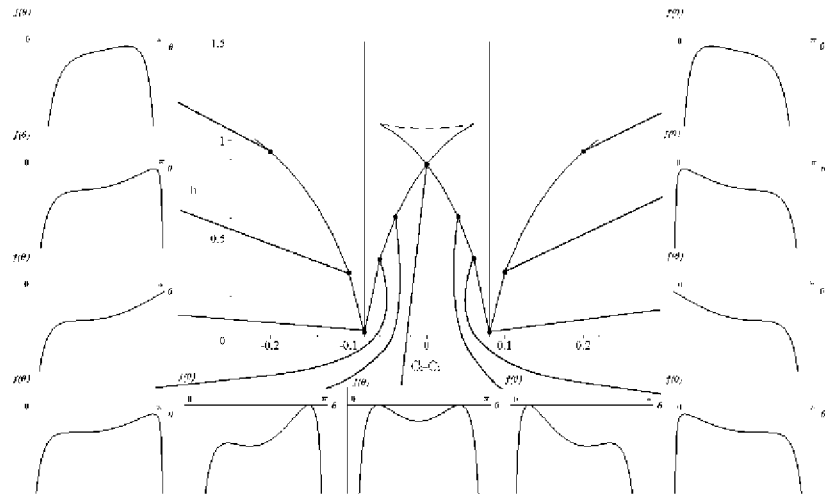


Fig. 6. Various types of the gyroscopic function for the section of the surface of regular precessions by plane $C_1 + C_2 = 0.08$.

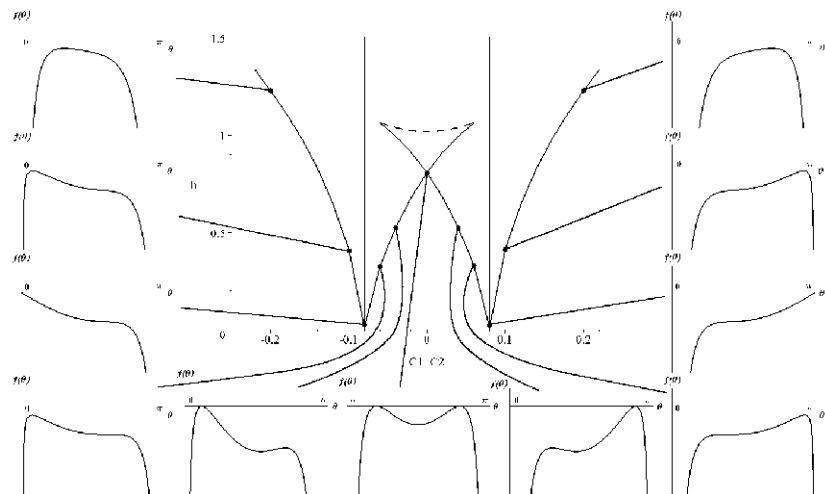


Fig. 7. Various types of the gyroscopic function for the section of the surface of regular precessions by plane $C_1 - C_2 = 0.08$.

3.2. The qualitative analysis of motion of apexes

The behavior of angles of proper rotation φ and precessions ψ that together with θ determined the motion of apexes is defined by relations (2.25). The important feature in this problem is the *two-frequency* behavior of each of these angles. That is not usual for integrable systems. For example, for the Kovalevskaya top the angle $\psi(t)$ is defined by three frequencies [3]. In this case such phenomena is connected to the existence of two methods of reduction with respect to the symmetries of system (2.12), (2.15).

From the geometrical point of view whole space of variables $\mathbf{M}, \alpha, \beta, \gamma$ is foliated on three-dimensional tori defined as the joint level surfaces of the integrals C_1, C_2, h and the geometrical integrals. The motion represents a winding of the three-dimensional torus with frequencies $\omega_\theta, \omega_\varphi, \omega_\psi$ [24]. (For the reduced systems (2.12) and (2.15) the corresponding tori are two-dimensional.)

Since the frequencies depend only on constants of the first integrals, all motion on the torus have the identical frequency that not evident for nonholonomic systems. Even for the integrable

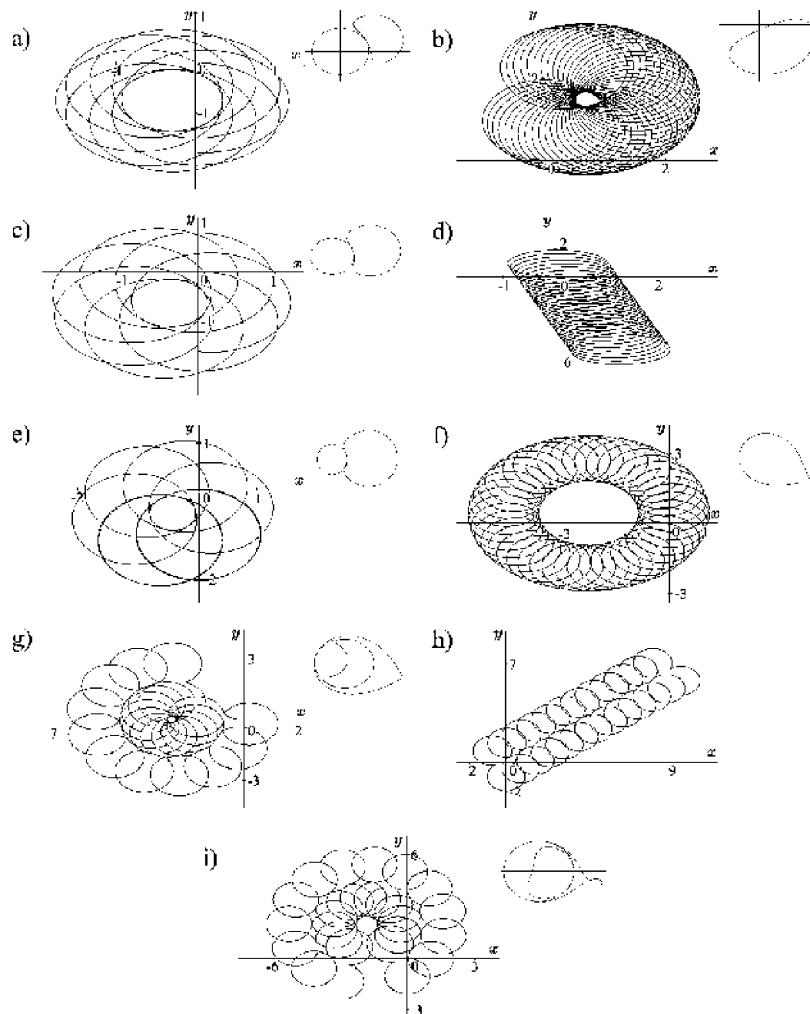


Fig. 8. Trajectories of the point of contact of the disk in the absolute space at various values of the integral of energy. Parameters of the system correspond to figure a). The closed trajectories in the frame of references rotating with the angular velocity ω_ψ (see explanations in the text) are presented in the right upper corner of each figure (except the case of infinite motions). In figures a) and b) we present various types of motion of the disk at the energy $h = 0.86$. Figures c) and d) correspond to the energy $h = 0.92217$ when one of the motions becomes resonant ($\omega_\theta = \omega_\psi^{(2)}$) and the secular drift (fig. d) is observed. The increase of the energy in figures e) and f) to $h = 0.961$ makes both types of motion bounded again. In figure g) the motion of the disk is presented at $h = 1.1$ after merging of two domains of possible motions corresponding to various types of motion. The infinite motion in figure h) corresponds to the resonance $\omega_\psi = 2\omega_\theta$ at the energy $h = 1.18169$. In figure i) the motion of the point of contact of the disk is presented after the further increase of the energy up to $h = 1.4$.

nonholonomic systems on two-dimensional tori there is a non-uniform rectilinear motion and, generally speaking, the intermixing is possible (see paper [4]).

Practically these arguments prove that the given system is Hamiltonian one in the analytical sense (though the Hamiltonian function can be a different from the energy (2.9) [24]. Moreover, taking into account only the analytical point of view we can say that near the nonsingular torus the system becomes the Hamiltonian one by the infinite number of methods [16].

REMARK 5. N. K. Moschuk in [25] observed a related phenomena studying the nonholonomic Chaplygin system possessing some number of the linear with respect to velocities first integrals.

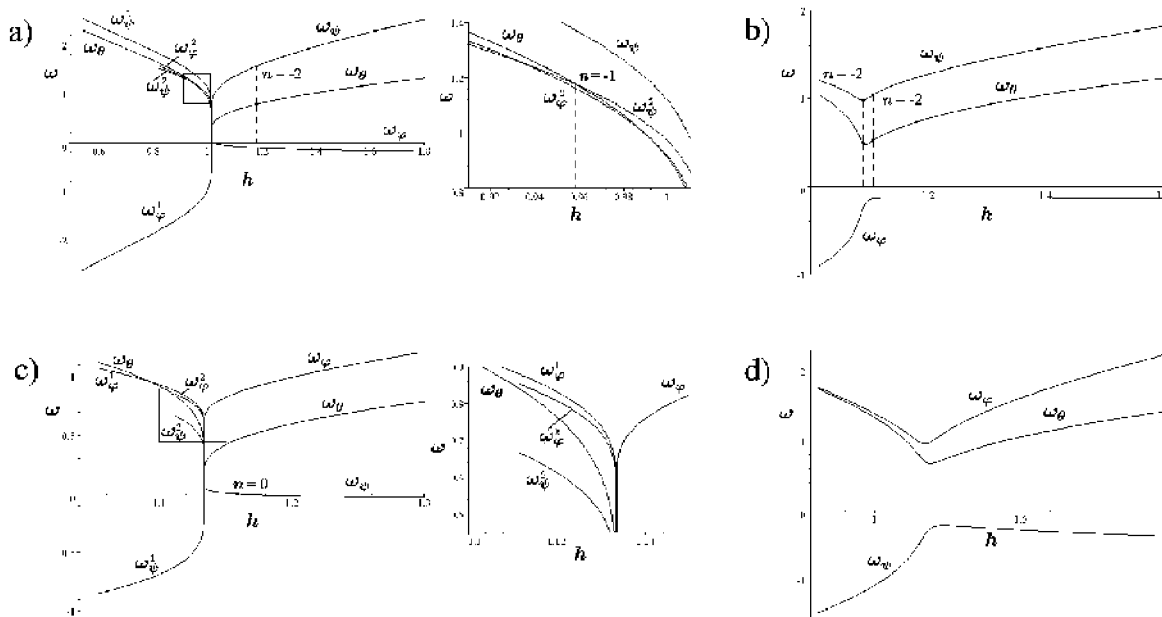


Fig. 9. Dependencies of frequencies ω_θ , ω_ψ and ω_φ from the energy at $I_1 = \frac{1}{4}$, $I_3 = \frac{1}{2}$, $R = 1$, $m = 1$, $g = 1$, and various values of integrals C_1 , C_2 . In figures a) and c) the areas marked by the rectangles are separately presented in the increased scale. In the field of energies where two different types of motions are possible in the absolute space we denote the frequencies by ω_ψ^1 , ω_φ^1 and ω_ψ^2 , ω_φ^2 . The resonance energies are marked on the graphs by thick dots. The orders of the resonances are indicated near the dots. The values of integrals for the dependencies presented here are: a) $C_1 = 0.04$, $C_2 = -0.02$; b) $C_1 = 0.09$, $C_2 = -0.07$; c) $C_1 = 0.065$, $C_2 = 0.055$; d) $C_1 = 0.09$, $C_2 = 0.03$.

At the same time the existence of a natural (algebraic) Poisson structure with a Hamiltonian defined by the energy (2.9) remains an open problem. A. V. Borisov and I. S. Mamaev show that the reduced system (2.19) is the Hamiltonian one with some algebraic nonlinear bracket (see [5]), however the possibility of its lifting on the systems (2.12) and (2.15) is still not investigated.

3.3. The analysis of motion of the point of contact.

For the analysis of motion of the point of contact we decompose the velocity (2.26) in the Fourier series with respect to time. Then from (2.25) we get

$$\dot{Z} = \sum_{n \in \mathbb{Z}} v_n e^{i(\omega_\psi + n\omega_\theta)t}.$$

Integrating with respect to time we obtain

$$Z(t) = Z_0 + e^{i\omega_\psi t} \sum_{n \in \mathbb{Z}} \frac{v_n}{i(\omega_\psi + n\omega_\theta)} e^{in\omega_\theta t}.$$

Thus, if at $\omega_\psi + n\omega_\theta \neq 0$ we use the frame of references rotating about the point Z_0 with the angular velocity ω_ψ , then the point of contact traces some closed curve (see [24], [20]). Various types of such closed curves and trajectories corresponding to them in the fixed space are presented in figure 8.

At the resonance $\omega_\psi + n\omega_\theta = 0$ we observe the secular drift of the point of contact. Graphs of frequencies $\omega_\psi(h)$, $\omega_\theta(h)$, $\omega_\varphi(h)$ at the fixed values of integrals C_1 , C_2 are presented in fig. 9. They show that the relation $\omega_\psi + n\omega_\theta = 0$ can be fulfilled both in the case of existence of one and of three regular precessions. And at the same energy some initial conditions lead to a secular drift while the others are not (see fig. 8). Since all frequencies depend only on the values of the first integrals the

relation $\omega_\psi + n\omega_\theta = 0$ define in three-dimensional space of integrals some two-dimensional manifold corresponding to the infinite trajectories of the disk.

Thus, for almost all initial conditions (except the indicated manifold) all trajectories of the disk are bounded.

We can consider this result to be opposite to the one obtained from research of dynamics of the point of contact for the Chaplygin ball on a horizontal plane (see [12]) where the majority of trajectories, on the contrary, were unbounded.

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